Nonlinear Unemployment Effects of the Inflation Tax

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Introduction

- Recent policy discussions about raising the inflation target or adopting average inflation targeting to avoid the "ZLB".
- Negative level effects of inflation tax on output are well known.
- Empirical literature on threshold effects of inflation on growth:
  - Sarel (1996), Bruno and Easterly (1998), Khan and Senhadji (2001), Drukker et al. (2005), and Kremer et al. (2013).
- We explore the possibility of nonlinear and state-dependent effects of the inflation tax on unemployment, output and welfare.
- We answer this question empirically and quantitatively.

What we do

- We document three novel stylized facts about inflation and unemployment in OECD countries:
    1. A positive long-run relationship between anticipated inflation and unemployment.
    2. A positive correlation between anticipated inflation and unemployment volatility.
    3. The long-run inflation-unemployment relationship is stronger when unemployment is higher.

- We show that these correlations arise in a standard monetary search model with two shocks – productivity and monetary – and frictions in labor and goods markets.

What we do

- Inflation tax lowers the surplus from a worker-firm match, in turn making it sensitive to productivity shocks or to further increases in inflation.
- We calibrate the model to match the US postwar labor market and monetary data and show that it is consistent with observed cross-country correlations.
- The model implies that the welfare cost of inflation is nonlinear in the level of inflation and is amplified by the presence of aggregate shocks.

Related macro literature

- Labor search: Shimer (2005), Hagedorn and Manovskii (2008), Hall and Milgrom (2008), Ljungqvist and Sargent (2017), Petrosky-Nadeau et al. (2018), and Petrosky-Nadeau and Zhang (2020)
- Labor search with liquidity: Berentsen et al. (2011), Gomis-Porqueras et al. (2013), Rocheteau and Rodriguez-Lopez (2014), Bethune et al. (2015), and Gomis-Porqueras et al. (2020).

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Data

- Quarterly panel data on 35 OECD countries (Main Economic Indicators database).
- Data on long-term nominal interest rates (10y government bonds) and unemployment rates (harmonized).
- We use the long-term nominal interest rate as a proxy for anticipated inflation.
- We focus on the trend (low frequency) component of each series (HP filter and 5y moving averages).

Stylized fact n° 1

- Is there a long-run relationship between unemployment and anticipated inflation?
- Regress trend unemployment on the trend long-term interest rate.
- Pooled OLS regression:

$$\bar{u}_{jt} = \alpha + \beta \bar{\iota}_{jt} + \varepsilon_{jt},$$

- Fixed-effects panel regression:

$$\bar{u}_{jt} = \alpha + \beta \bar{\iota}_{jt} + \gamma_j + \delta_t + \varepsilon_{jt},$$

Stylized fact n° 1

	Trend unemployment (HP filter)			
	(1)	(2)	(3)	(4)
Constant	5.771***	6.036***	3.739***	3.498***
	(0.618)	(0.362)	(1.366)	(1.209)
Trend long-term rate (HP filter)	0.351***	0.301***	0.727**	0.772***
	(0.091)	(0.062)	(0.288)	(0.224)
Observations	4,007	4,007	4,007	4,007
R^2	0.086	0.140	0.121	0.135
F-Statistic	377.98***	646.61***	515.56***	581.55***
Country fixed effects	No	Yes	No	Yes
Time fixed effects	No	No	Yes	Yes
Clustered errors (country level)	Yes	Yes	Yes	Yes

*p<0.1; **p<0.05; ***p<0.01

▸ in log

Stylized fact n° 1

	Trend unemployment (5y moving average)			
	(1)	(2)	(3)	(4)
Constant	5.837***	6.067***	3.005***	2.837***
	(0.597)	(0.385)	(1.302)	(1.029)
Trend long-term rate (5y moving average)	0.366***	0.324***	0.884***	0.915***
	(0.112)	(0.071)	(0.269)	(0.188)
Observations	3,262	3,262	3,262	3,262
R^2	0.083	0.142	0.167	0.200
F-Statistic	295.68***	532.55***	517.21***	744.80***
Country fixed effects	No	Yes	No	Yes
Time fixed effects	No	No	Yes	Yes
Clustered errors (country level)	Yes	Yes	Yes	Yes

*p<0.1; **p<0.05; ***p<0.01

▸ in log

Stylized fact n° 2

- Is there a relationship between anticipated inflation and unemployment volatility?
- Regress unemployment volatility on trend long-term interest rate.
- Unemployment volatility is measured as the standard deviation of cyclical log unemployment over a 5y moving window.
- Pooled OLS regression:

$$\sigma_{u_{jt}} = \alpha + \beta \bar{\iota}_{jt} + \varepsilon_{jt},$$

- Fixed effects panel regression:

$$\sigma_{u_{jt}} = \alpha + \beta \bar{\iota}_{jt} + \gamma_j + \delta_t + \varepsilon_{jt},$$

Stylized fact n° 2

	log unemployment volatility (HP filter)			
	(1)	(2)	(3)	(4)
Constant	0.058***	0.052***	0.060***	0.031
	(0.005)	(0.007)	(0.011)	(0.023)
Trend long-term rate (HP filter)	0.005***	0.006***	0.005*	0.010**
	(0.001)	(0.001)	(0.002)	(0.004)
Observations	3,616	3,616	3,616	3,616
R^2	0.079	0.115	0.031	0.062
F-Statistic	310.07***	463.69***	109.18***	221.79***
Country fixed effects	No	Yes	No	Yes
Time fixed effects	No	No	Yes	Yes
Clustered errors (country level)	Yes	Yes	Yes	Yes

*p<0.1; **p<0.05; ***p<0.01

▸ in level

Stylized fact n° 2

	log unemployment volatility (5y moving average)			
	(1)	(2)	(3)	(4)
Constant	0.099***	0.085***	0.064***	0.007
	(0.013)	(0.019)	(0.023)	(0.048)
Trend long-term rate (5y moving average)	0.008***	0.011***	0.015***	0.026***
	(0.003)	(0.004)	(0.005)	(0.009)
Observations	2,882	2,882	2,882	2,882
R^2	0.065	0.113	0.078	0.097
F-Statistic	201.77***	364.07***	224.29***	282.66***
Country fixed effects	No	Yes	No	Yes
Time fixed effects	No	No	Yes	Yes
Clustered errors (country level)	Yes	Yes	Yes	Yes

*p<0.1; **p<0.05; ***p<0.01

▸ in level

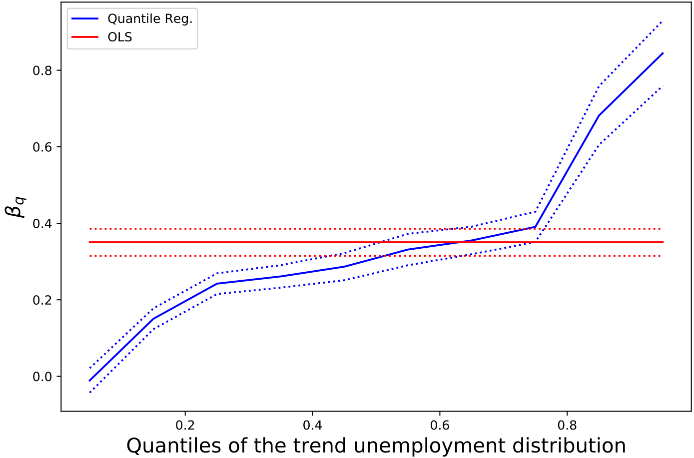
Stylized fact n° 3

- Does the long-run inflation-unemployment relationship vary in the level of unemployment?
- Regress trend unemployment on trend long-term interest rate for different quantiles of unemployment.
- Quantile regression approximates the conditional quantile function at quantile q by a linear relationship.
- Pooled quantile regression:

$$\mathcal{Q}_q(\bar{u}_{jt}|\bar{\iota}_{jt}) = \alpha_q + \beta_q \bar{\iota}_{jt} + \varepsilon_{qjt}.$$

Stylized fact n° 3

Figure: Quantile regressions of $\bar{u}$ on $\bar{\iota}$ (HP filter).



5y average | log 5y average | log HP filter

Content

Model overview

- Standard monetary search model (Berentsen et al., 2011).
- Labor market frictions give rise to equilibrium unemployment (Pissarides, 2000).
- Goods market frictions generate a transaction demand for money (Lagos and Wright, 2005).
- Stochastic productivity y_t and nominal interest rate ι_t.

Environment

- Discrete time. Infinitely lived agents. Discounting factor β.
- Unit measure of workers, either employed (e) or unemployed (u).
- Large number of firms with free entry.
- 3 sequential markets take place in each period:
  - Decentralized labor market (LM);
  - Decentralized goods markets (DM);
  - Centralized goods market (CM).
- Two perishable goods: CM good y (numeraire) and DM good x.

Environment

- Aggregate state: $\Omega_t = \{n_t, y_t, \iota_t\}$
- Productivity and monetary shocks are realized at the beginning of the CM.
- Fisher equation: $\iota_t = (1 + \pi_t)/\beta - 1$ where π_t is inflation.
- Fiat money supply M_t grows stochastically via lump-sum transfers $T(\iota_t)$ in the CM.

Preferences and technology

- Worker preferences:

$$\sum_{t=0}^{\infty} \beta^t \left(\mathrm{u}\left(x_t\right) + c_t\right)$$

where $c_t =$ CM good, $x_t =$ DM good.

- Firm hires worker to produce quantity y of CM goods.
- Firm can produce x units of DM goods on-demand at cost $C(x)$.

Labor Market (LM)

- Random search and matching between vacancies and unemployed workers.
- LM tightness: $\theta_t = \frac{v_t}{1-n_t}$
- Matching probabilities: $f(\theta_t) = \theta_t q(\theta_t)$
- Exogenous job separation at rate δ.
- Law of motion for employment:

$$n_{t+1} = (1-\delta)\, n_t + f(\theta_t)\, (1-n_t)$$

LM value functions LM bargaining

Decentralized Goods Market (DM)

- Random matching between buyers/workers and firms.
- Buyers' matching probability: $\alpha(n_t)$.
- Sellers' matching probability: $\frac{\alpha(n_t)}{n_t}$.
- Informational frictions require the use of liquid assets for immediate settlement.
- Price setting: proportional bargaining (Kalai, 1977).

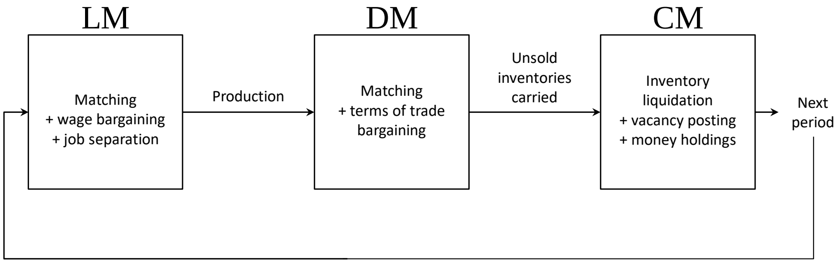
DM value functions DM bargaining

Centralized Goods and Settlement Market (CM)

- Frictionless Walrasian market.
- Firms liquidate inventories, pay wage w_t and distribute profits.
- Households consume x_t and decide on money holdings z_{t+1}.
- Central banks distributes lump-sum transfers $T(\iota_t)$.

▸ CM value functions

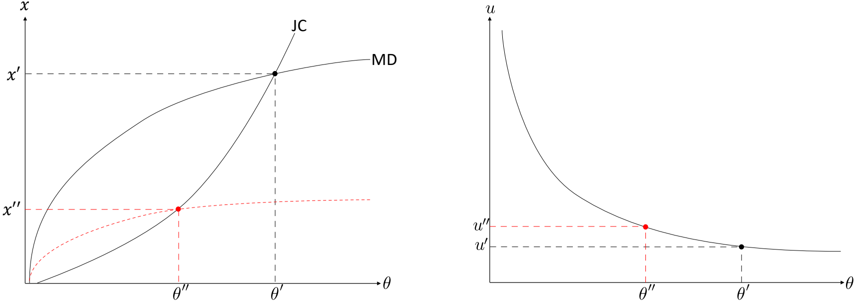
Timeline


LM
Matching
+ wage bargaining
+ job separation
Production
DM
Matching
+ terms of trade
bargaining
Unsold
inventories
carried
CM
Inventory
liquidation
+ vacancy posting
+ money holdings
Next
period

Unemployment and inflation tax

- Inflation matters for firm-worker match surplus through the DM.
- An increase in anticipated inflation increases unemployment:

$$\frac{dx}{d\iota} < 0 \Rightarrow \frac{d\theta}{d\iota} < 0 \Rightarrow \frac{du}{d\iota} > 0$$



Nonlinear inflation effects

- What about the nonlinear effects of inflation?

- Following Ljungqvist and Sargent (2017), we have:

$$\varepsilon_{\theta,y} = \left(1 - \frac{1}{\epsilon_{\Upsilon,\theta}} \frac{\mathcal{O}}{\mathcal{O}-b} \frac{\mathcal{P}}{\mathcal{O}} \varepsilon_{\mathcal{P},n} \varepsilon_{n,\theta}\right)^{-1} \frac{1}{\epsilon_{\Upsilon,\theta}} \frac{\mathcal{O}}{\mathcal{O}-b} \frac{y}{\mathcal{O}},$$

$$\varepsilon_{\theta,\iota} = \left(1 - \frac{1}{\epsilon_{\Upsilon,\theta}} \frac{\mathcal{O}}{\mathcal{O}-b} \frac{\mathcal{P}}{\mathcal{O}} \varepsilon_{\mathcal{P},n} \varepsilon_{n,\theta}\right)^{-1} \frac{1}{\epsilon_{\Upsilon,\theta}} \frac{\mathcal{O}}{\mathcal{O}-b} \frac{\mathcal{P}}{\mathcal{O}} \varepsilon_{\mathcal{P},\iota},$$

- Higher trend inflation amplifies unemployment responsiveness to both productivity and monetary shocks.
- Feedback effects through goods market frictions.

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Calibration

- Model is set to a monthly frequency.
- We calibrate the model to match post-war US data (January 1948 to December 2019).
- We match both monthly and quarterly moments.
- Model is solved globally and calibrated using Simulated Method of Moments.

Calibration - Stochastic processes

- Labor productivity shock:

$$\log y_{t+1} = (1 - \rho_y) \log \bar{y} + \rho_y \log y_t + \varepsilon_{y,t+1}$$

- Nominal interest rate shock:
  - We decompose the shock into a trend and cycle components:

$$\iota_t = \bar{\iota}_t + \hat{\iota}_t$$

  - The cyclical component is modeled as stationary AR1 process:

$$\hat{\iota}_{t+1} = \rho_{\hat{\iota}} \hat{\iota}_t + \varepsilon_{\hat{\iota},t+1}$$

  - The trend component is modeled as a very persistent Markov chain with 5 states (transition probabilities estimated using ML).

Calibration - Functional forms

- LM matching function: $f(\theta) = \theta q(\theta) = \frac{\theta}{(1+\theta^{\chi})^{\frac{1}{\chi}}}$ (Den Haan et al., 2000).
- DM matching function: $\alpha(n) = \zeta \frac{n}{n+1}$
- DM utility: $u(x) = A\frac{x^{1-\gamma}}{1-\gamma}$
- DM cost:

$$C(x) = x$$

Calibration - External parameters

Parameter	Description	Value	Source
β	Discount factor	0.998	Data
δ	Job separation probability	0.025	Data
$\bar{y}$	Average labor productivity	1.00	Normalization
$\rho_{\hat{\imath}}$	Autocorr. of interest rate shocks	0.939	Data
$\sigma_{\varepsilon_{\hat{\imath}}}$	SD of interest rate shocks	0.0001	Data

Calibration - Simulated Method of Moments

- Vector of 10 parameters Θ.
- Vectors of 10 moments in the data μ and model $\mu_s(\Theta)$.
  - Model moments averaged over $S = 1'000$ simulations of length $T = 1'000$.
  - Burn first 133 observations to match length of data (867 months).
- Minimize the distance $G(\Theta) = \mu - \frac{1}{S}\sum_{s=1}^{S} \mu_s(\Theta)$:

$$\hat{\Theta} = \arg\min_{\Theta} G(\Theta)^T W^{-1} G(\Theta)$$

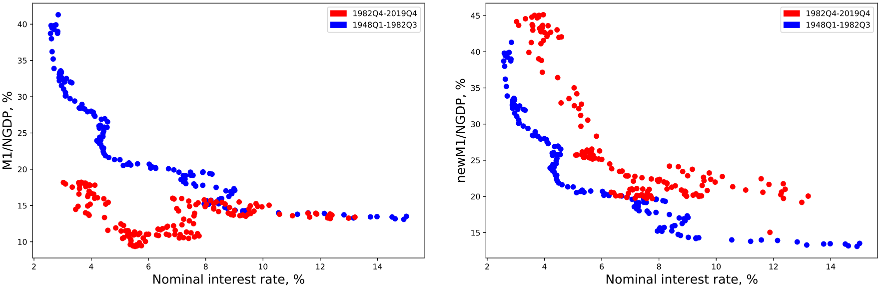
where W is a weighting matrix.

Calibration - Data

- Labor market data:
  - Unemployment rate: CPS, civilian population under 16.
  - Job vacancy rate: Barnichon data and JOLTS.
  - Job separation rate: constructed using short-term unemployment.
  - Job finding rate: constructed using short-term unemployment.
  - Labor productivity: BLS non-farm real output per person.
  - Real wage: labor productivity x BLS labor income share.
- Monetary data:
  - Monetary aggregate: M1+MMDA (Rasche, 1987; Lucas and Nicolini, 2015).
  - Interest rate: Moody's AAA long-term corporate bond index.
  - Nominal GDP.
  - CPI inflation.
  - Markup: data from De Loecker et al. (2020).

Calibration - Data

Figure: Measuring money demand: M1 v. M1+MMDA

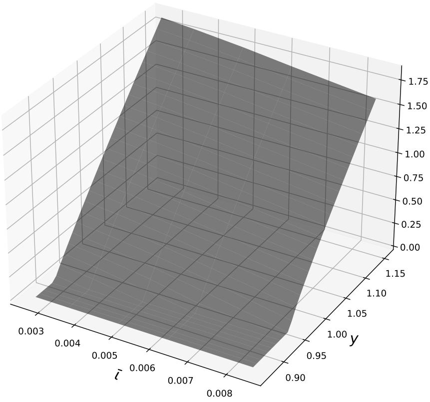


Calibration - Results

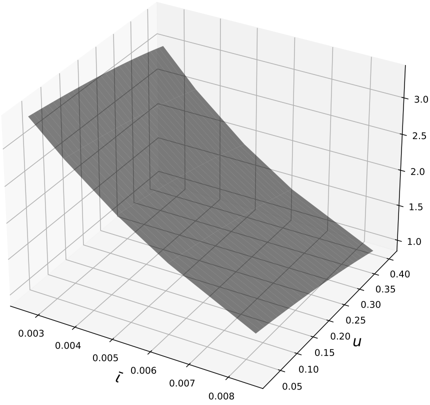
Table: SMM calibrated parameters

Parameter	Description	Value	Moment	Frequency	Data	Model
κ	Vacancy cost	1.471	Average θ	Monthly	0.634	0.634
b	Flow value of unemployment	0.990	Unemployment volatility	Quarterly	0.138	0.138
χ	Parameter of the LM matching fun.	1.269	Average JFP	Monthly	0.430	0.430
ξ	Worker bargaining weight	0.035	Elast. of wage to labor prod.	Quarterly	0.470	0.470
ρ_y	Persistence parameter of y_t process	0.967	Autocorr. of labor productivity	Quarterly	0.758	0.760
σ_y	Volatility parameter of y_t process	0.007	SD of labor productivity	Quarterly	0.013	0.013
A	Level parameter of DM utility	1.421	Average money demand	Quarterly	25.73%	25.72%
γ	Curvature parameter of DM utility	0.217	Elast. of money demand to ι	Quarterly	-0.594	-0.594
ζ	Parameter of the DM matching fun.	0.204	Elast. of u to ι	Monthly	0.297	0.297
φ	Buyer bargaining weight	0.320	Average price markup	Monthly	36.00%	36.00%

Policy Functions



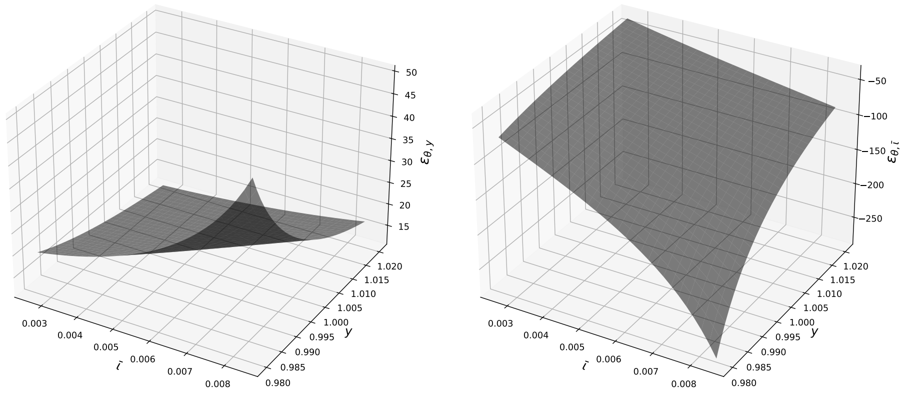
(a) $\theta(\Omega)$ in $(\bar{\iota}, y)$ space



(b) $q(\Omega)$ in $(\bar{\iota}, u)$ space

Figure: Policy functions of the calibrated model.

Steady State Elasticities



(a) Elasticity of θ wrt. y, $\varepsilon_{\theta,y}$

(b) Semi-elasticity of θ wrt. ι, $\varepsilon_{\theta,\iota}$

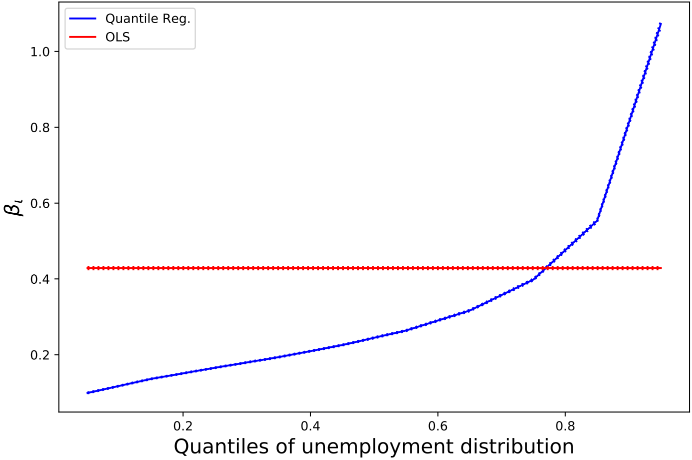
Figure: Steady state elasticities of θ in the calibrated model.

Business Cycle Statistics

		u	v	θ	$\mathcal{O}$
Quarterly US data, 1948-2019					
Standard deviation		0.138	0.137	0.257	0.013
Autocorrelation		0.895	0.902	0.903	0.758
Correlation matrix	u	1	-0.900	-0.950	-0.231
	v	-	1	0.982	0.363
	θ	-	-	1	0.296
	$\mathcal{O}$	-	-	-	1
Model simulations					
Standard deviation		0.137	0.627	0.740	0.013
Autocorrelation		0.843	0.431	0.636	0.760
Correlation matrix	u	1	-0.559	-0.792	-0.851
	v	-	1	0.903	0.643
	θ	-	-	1	0.758
	$\mathcal{O}$	-	-	-	1

Linear and Quantile Regressions

Figure: Linear and Quantile regressions of trend u on ι using simulated data



Unemployment Volatility Regression

	Unemployment volatility (5y rol. wind. SD)
	(1)
Constant	0.031***
	(0.000)
Trend long-term rate (HP filter)	0.013***
	(0.000)
Observations	269,000
R^2	0.182

*p<0.1; **p<0.05; ***p<0.01

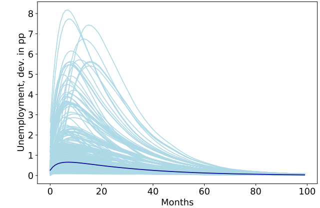
Generalized Impulse Response Functions

- State-dependent reaction to shocks.
- Generalized impulse response function (Koop et al., 1996):

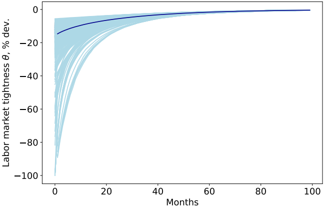
$$GIRF_Y(k, \varepsilon_t, \Omega_t) = \mathbb{E}[Y_{t+k}|\varepsilon_t, \Omega_t = \omega_t] - \mathbb{E}[Y_{t+k}|\Omega_t = \omega_t],$$

where ω_t is the state of economy at the beginning of period t.

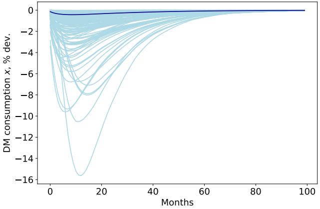
Generalized Impulse Response Functions



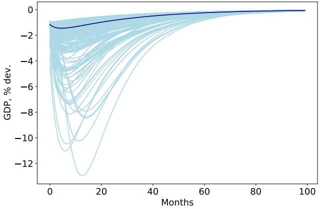
(a) Unemployment



(b) LM tightness θ



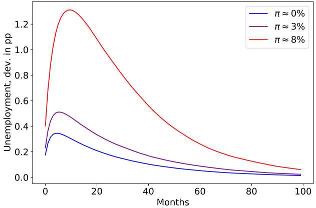
(c) DM consumption x



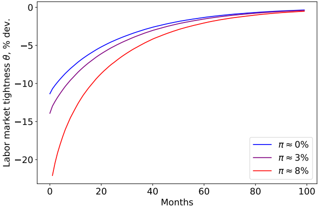
(d) Total output

Figure: GIRFs following a negative productivity shock

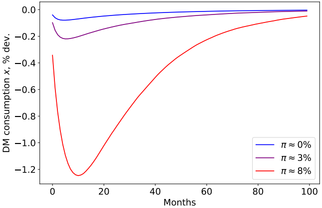
Generalized Impulse Response Functions



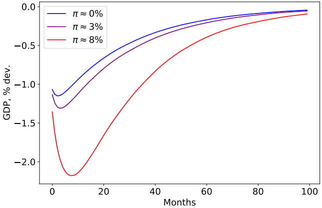
(a) Unemployment



(b) LM tightness θ



(c) DM consumption x



(d) Total output

Figure: Average GIRFs following a negative productivity shock conditional on trend inflation

Welfare Cost of Inflation

1. Simulate the model with cyclical shocks under different levels of trend inflation.
2. Compute average welfare for each trend inflation level:

$$\mathcal{W}(\Omega_t) = \alpha(n_t)[u(x_t) - c(x_t)] + n_t y_t + (1 - n_t)b - \kappa v_t/\beta.$$

Annual inflation rate	**Implied interest rate**	**Flow welfare level**	**Difference with FR**
-2.75%	0.00%	1.084	-
0.00%	2.82%	1.080	-0.37%
5.00%	7.97%	1.061	-2.13%
10.00%	13.11%	1.035	-4.52%

Table: Welfare cost of inflation in baseline economy

Welfare Cost of Inflation

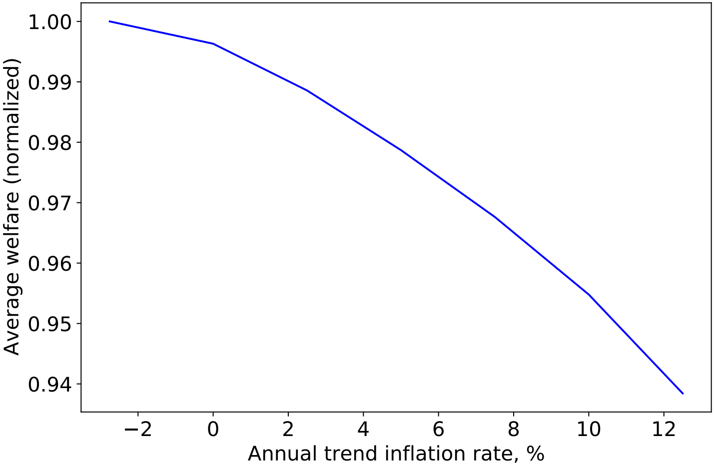


Figure: Nonlinearity of the welfare cost of trend inflation.

Welfare Cost of Inflation

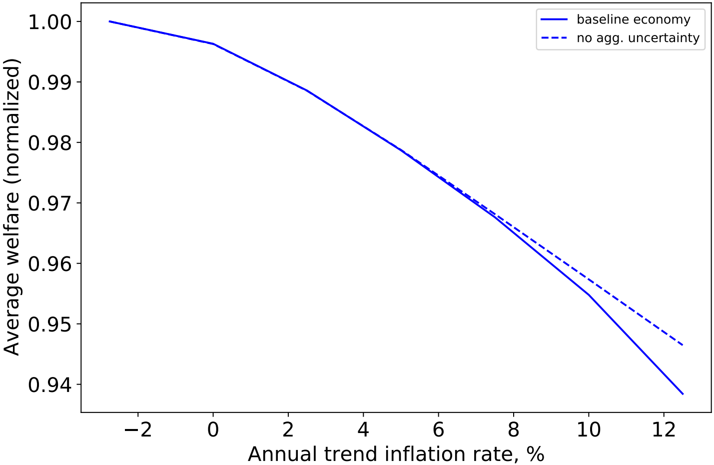


Figure: Contribution of aggregate uncertainty to the cost of inflation.

Content

Conclusion

- Evidence of a positive and nonlinear long-run relationship between anticipated inflation and unemployment.
- A standard monetary search model with productivity and interest rate shocks can replicate these facts.
- The nonlinear unemployment effects amplify substantially the welfare cost of inflation.
- The business cycle is not invariant to the long-run inflation target.

Stylized fact n° 1

	Trend log unemployment (HP filter)			
	(1)	(2)	(3)	(4)
Constant	1.707***	1.755***	1.555***	1.705***
	(0.075)	(0.050)	(0.175)	(0.114)
Trend long-term rate (HP filter)	0.039***	0.031***	0.067**	0.039*
	(0.011)	(0.009)	(0.033)	(0.021)
Observations	4,007	4,007	4,007	4,007
R^2	0.072	0.090	0.072	0.024
F-Statistic	312.93***	395.14***	291.58***	92.49***
Country fixed effects	No	Yes	No	Yes
Time fixed effects	No	No	Yes	Yes
Clustered errors (country level)	Yes	Yes	Yes	Yes

*p<0.1; **p<0.05; ***p<0.01

◂ in level

Stylized fact n° 1

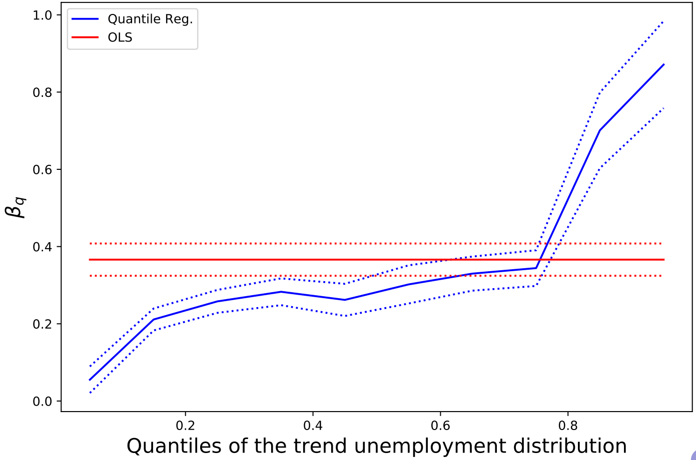
	Trend log unemployment (5y moving average)			
	(1)	(2)	(3)	(4)
Constant	1.728***	1.776***	1.493***	1.663***
	(0.082)	(0.045)	(0.156)	(0.098)
Trend long-term rate (5y moving average)	0.041***	0.032***	0.084***	0.053***
	(0.012)	(0.008)	(0.028)	(0.018)
Observations	3,262	3,262	3,262	3,262
R^2	0.075	0.101	0.110	0.052
F-Statistic	263.02***	364.35***	374.11***	164.11***
Country fixed effects	No	Yes	No	Yes
Time fixed effects	No	No	Yes	Yes
Clustered errors (country level)	Yes	Yes	Yes	Yes

*p<0.1; **p<0.05; ***p<0.01

◀ in level

Stylized fact n°2

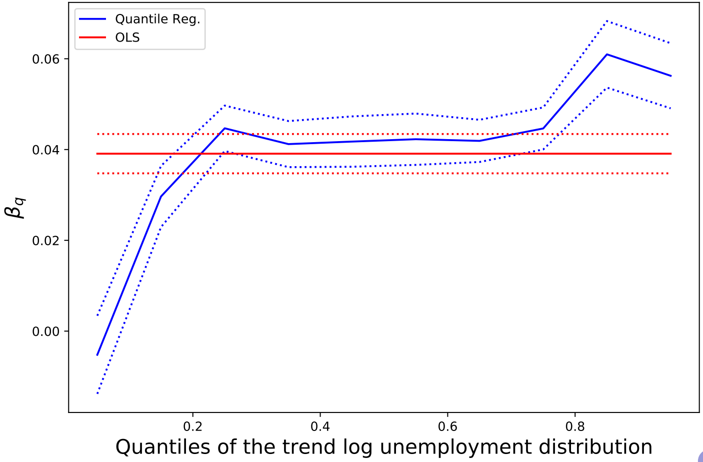
Figure: Quantile regression of u on i (5y average)



◂ level HP filter

Stylized fact n°2

Figure: Quantile regression of log u on i (HP filter)



level HP filter

Stylized fact n°2

Figure: Quantile regression of log u on i (5y average)

◂ level HP filter

Stylized fact n° 3

	Unemployment volatility (5y moving window SD)			
	(1)	(2)	(3)	(4)
Constant	0.390***	0.354***	0.222*	-0.024
	(0.050)	(0.053)	(0.122)	(0.166)
Trend long-term rate (HP filter)	0.046***	0.053***	0.079***	0.128***
	(0.008)	(0.010)	(0.028)	(0.032)
Observations	3,616	3,616	3,616	3,616
R^2	0.077	0.132	0.090	0.135
F-Statistic	301.46***	544.71***	333.41***	519.35***
Country fixed effects	No	Yes	No	Yes
Time fixed effects	No	No	Yes	Yes
Clustered errors (country level)	Yes	Yes	Yes	Yes

*p<0.1; **p<0.05; ***p<0.01

in log

Stylized fact n° 3

	Unemployment volatility (5y moving window SD)			
	(1)	(2)	(3)	(4)
Constant	0.588***	0.523***	-0.234	-0.740**
	(0.142)	(0.129)	(0.288)	(0.357)
Trend long-term rate (5y average)	0.098***	0.110***	0.256***	0.354***
	(0.031)	(0.025)	(0.065)	(0.069)
Observations	2,882	2,882	2,882	2,882
R^2	0.079	0.139	0.196	0.216
F-Statistic	248.16***	460.15***	650.05***	721.46***
Country fixed effects	No	Yes	No	Yes
Time fixed effects	No	No	Yes	Yes
Clustered errors (country level)	Yes	Yes	Yes	Yes

*p<0.1; **p<0.05; ***p<0.01

in log

LM - Workers

- Employed worker with liquid assets z:

$$V^e_{LM}(z_t, \Omega_t) = (1-\delta)\, V^e_{DM}(z_t, w_t, \Omega_t) + \delta\, V^u_{DM}(z_t, \Omega_t)$$

- Unemployed worker with liquid assets z:

$$V^u_{LM}(z_t, \Omega_t) = f(\theta_t)\, V^e_{DM}(z_t, w_t, \Omega_t) + (1 - f(\theta))\, V^u_{DM}(z_t, \Omega_t)$$

◂ LM overview

LM - Firms

- Firm with a worker:

$$J^{e}_{LM}(\Omega_t) = (1-\delta)\, J^{e}_{DM}(w_t, \Omega_t)$$

- Firm without a worker:

$$J^{v}_{LM}(\Omega_t) = q(\theta_t)\, J^{e}_{DM}(w_t, \Omega_t)$$

LM overview

DM - Workers

- Employed worker with liquid assets z_t:

$$
\begin{aligned}
V^e_{DM}\left(z_t, w_t, \Omega_t\right) =& \alpha\left(n_{t+1}\right)\Big[\mathrm{u}\left(x\left(z_t\right)\right) \\
&+\mathbb{E}V^e_{CM}\left(z_t - d\left(z_t\right) + T\left(\iota_t\right) + w_t, \Omega_{t+1}\right)\Big] \\
&+\left(1-\alpha\left(n_{t+1}\right)\right)\mathbb{E}V^e_{CM}\left(z_t + T\left(\iota_t\right) + w_t, \Omega_{t+1}\right)
\end{aligned}
$$

- Unemployed worker with liquid assets z_t:

$$
\begin{aligned}
V^u_{DM}\left(z_t, \Omega_t\right) =& \alpha\left(n_{t+1}\right)\Big[\mathrm{u}\left(x\left(z_t\right)\right) \\
&+\mathbb{E}V^u_{CM}\left(z_t - d\left(z_t\right) + T\left(\iota_t\right) + b, \Omega_{t+1}\right)\Big] \\
&+\left(1-\alpha\left(n_{t+1}\right)\right)\mathbb{E}V^u_{CM}\left(z_t + T\left(\iota_t\right) + b, \Omega_{t+1}\right)
\end{aligned}
$$

◂ DM overview

DM - Firms

- Firm with a worker produces and sells its output, getting

$$J^e_{DM}\left(w_t, \Omega_t\right) = \frac{\alpha\left(n_{t+1}\right)}{n_{t+1}} \mathbb{E} J^e_{CM}\left(y_t - C(x_t; y_t) + d_t, w_t, \Omega_{t+1}\right) + \left(1 - \frac{\alpha\left(n_{t+1}\right)}{n_{t+1}}\right) \mathbb{E} J^e_{CM}\left(y_t, w_t, \Omega_{t+1}\right)$$

DM overview

CM - Workers

- Worker with employment status $j \in \{e, u\}$ and liquid assets z:

$$V^{j}_{CM}(z_t, \Omega_{t+1}) = \max_{c_t, z_{t+1}} c_t + \beta V^{j}_{LM}(z_{t+1}, \Omega_{t+1})$$

subject to

$$c_t + (1 + \pi_t) z_{t+1} = z_t$$

CM overview

CM - Firms

- Firm with a worker sells its inventories o and pays the wage w:

$$J^{e}_{CM}\left(o_t, w_t, \Omega_{t+1}\right) = o_t - w_t + \beta J^{e}_{LM}\left(\Omega_{t+1}\right)$$

- A firm without a worker decides whether to post a vacancy at cost κ:

$$J^{v}_{CM}\left(\Omega_{t+1}\right) = \max\left\{0, -\kappa + \beta J^{v}_{LM}\left(\Omega_{t+1}\right)\right\}$$

◂ CM overview

DM bargaining

- Kalai (1977) bargaining solution:

$$\max_{x_t, d_t} \mathrm{u}(x_t) - d_t$$

subject to

$$\mathrm{u}(x_t) - d_t = \frac{\varphi}{1-\varphi}\left[d_t - C(x_t; y_t)\right] \quad ; \quad d_t \leq z_t$$

- Solution is a pair (x_t, d_t) that satisfies

$$x_t = \min\left\{x^*(y_t), \mathrm{g}^{-1}(z_t; y_t)\right\}$$
$$d_t = \min\left\{\mathrm{g}(x^*(y_t); y_t), z_t\right\}$$

where $\mathrm{g}(x_t; y_t) = (1-\varphi)\,\mathrm{u}(x_t) + \varphi C(x_t; y_t)$
and $x^*(y_t)$ solves $\mathrm{u}'(x_t) - C_x(x_t; y_t) = 0$

◂ DM overview

Optimal choice of real balances

- Given the bargaining solution we have

$$\frac{\partial V^j_{LM}}{\partial z_t} = 1 + \alpha\left(n_{t+1}\right)\max\left\{0, \frac{\mathrm{u}'\left(x_t\right)}{\mathrm{g}'\left(x_t; y_t\right)} - 1\right\}$$

▸ DM bargaining

- In the CM, the first-order condition for z is

$$1 + \iota_t = \frac{\partial V^j_{LM}}{\partial z_{t+1}}$$

- Combining the above, we get

$$\mathrm{u}'\left(x_t\right) = \left(1 + \frac{\iota_t}{\alpha\left(n_{t+1}\right)}\right)\mathrm{g}'\left(x_t; y_t\right)$$

and $z_t = \mathrm{g}\left(x_t; y_t\right)$.

LM bargaining

- Worker's surplus from being employed at wage w is

$$S^e_{DM}(w_t, \Omega_t) \equiv V^e_{DM}(z_t, w_t, \Omega_t) - V^u_{DM}(z_t, \Omega_t)$$
$$= w_t - b + \beta \mathbb{E}\left(1 - \delta - f(\theta(\Omega_{t+1}))\right) S^e_{DM}(w_{t+1}, \Omega_{t+1})$$

- The firm's surplus from having a worker at wage w is

$$J^e_{DM}(w_t, \Omega_t) = \mathcal{O}(\Omega_t) - w_t + \beta(1-\delta)\mathbb{E}J^e_{DM}(w_{t+1}, \Omega_{t+1})$$

where $\mathcal{O}(\Omega_t) = y_t + \frac{\alpha(n_{t+1})}{n_{t+1}}(d_t - C(x_t; y_t))$

◂ Match surplus

LM bargaining

- The surplus from an employment match is

$$\mathcal{S}\left(\Omega_t\right) = S^e_{DM}\left(w_t, \Omega_t\right) + J^e_{DM}\left(w_t, \Omega_t\right)$$

- Wage $w_t = w(\Omega_t)$ is determined by Nash bargaining such that

$$S^e_{DM}\left(w_t, \Omega_t\right) = \xi \mathcal{S}\left(\Omega_t\right)$$

- Wage equation:

$$w\left(\Omega_t\right) = \xi \mathcal{O}\left(\Omega_t\right) + \left(1 - \xi\right) b + \mathbb{E} \xi \kappa \theta \left(\Omega_{t+1}\right)$$

◀ Firm and worker surplus equations

Job surplus and free entry

- Recursive formulation:

$$\mathcal{S}(\Omega_t) = \mathcal{O}(\Omega_t) - b + \beta\mathbb{E}\left(1 - \delta - \xi f(\theta(\Omega_{t+1}))\right)\mathcal{S}(\Omega_{t+1})$$

- $\theta_t = \theta(\Omega_t)$ is determined by the free entry condition

$$\kappa = \beta q(\theta_t)(1 - \xi)\mathcal{S}(\Omega_t)$$

Equilibrium

The equilibrium consists of functions $x(\Omega), \mathcal{O}(\Omega), \mathcal{S}(\Omega), \theta(\Omega), w(\Omega), n_{t+1}(\Omega)$ such that

1. $x(\Omega)$ solves the optimal choice of real balances.
2. Output $\mathcal{O}(\Omega)$ is given by DM bargaining solution.
3. Surplus from a job match $\mathcal{S}(\Omega)$ satisfies its Bellman equation.
4. Free entry condition determines $\theta(\Omega)$
5. The wage $w(\Omega)$ satisfies LM bargaining solution.
6. Employment $n_{t+1}(\Omega)$ is given by it law of motion.